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Extension of the Buchalla–Safir bound

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Abstract. I provide a simple derivation of the Buchalla–Safir bound on γ. I generalize it to the case where an upper bound on the phase of the penguin pollution is assumed. I apply the Buchalla–Safir bound, and its generalization, to the most recent data on CP violation in $B \to \pi^+\pi^-$.

1 Introduction

CP violation in B_d^0–$\bar{B}_d^0$ mixing and in the decays of those mesons to $\pi^+\pi^-$ is parametrized by

$$\lambda = \frac{q}{p}\,\frac{\bar{A}}{A}, \tag{1}$$

where q/p relates to B_d^0–$\bar{B}_d^0$ mixing, A is the amplitude for $B_d^0 \to \pi^+\pi^-$, and $\bar{A}$ is the amplitude for $\bar{B}_d^0 \to \pi^+\pi^-$ [1]. Two CP-violating quantities can be measured:

$$S = \frac{2\,\mathrm{Im}\,\lambda}{1+|\lambda|^2}, \tag{2}$$

$$C = \frac{1-|\lambda|^2}{1+|\lambda|^2}. \tag{3}$$

Let

$$\frac{q}{p} = \exp\left(-2\mathrm{i}\tilde{\beta}\right). \tag{4}$$

In the standard model (SM), $\tilde{\beta} = \beta$ and the sine of 2β is measured [2] through CP violation in $B_d^0/\bar{B}_d^0 \to \psi K_S$:

$$\sin 2\beta = 0.736 \pm 0.049. \tag{5}$$

In the SM β must be smaller than $\pi/4$, hence $\cos 2\beta$ is assumed to be positive.

Together with (4), I shall assume that, as in the SM,

$$\frac{\bar{A}}{A} = \frac{\mathrm{e}^{-\mathrm{i}\gamma}+z}{\mathrm{e}^{\mathrm{i}\gamma}+z}, \tag{6}$$

where γ is another CP-violating phase, which one would like to be able to measure too. In the SM, $0 \le \gamma \le \pi - \beta$. The parameter z represents the "penguin pollution", an annoying contribution from penguin diagrams which we must somehow circumvent if we want to get at γ.

Buchalla and Safir (BS) [3] have found a solution to the following problem. Suppose that

(1) one has measured $\sin 2\tilde{\beta}$ and S,
(2) one has found that $S > -\sin 2\tilde{\beta}$,
(3) one assumes the validity of the SM, and
(4) one assumes that $\mathrm{Re}\,z > 0$.

Is it then possible to find a lower bound on γ stronger than $\gamma \ge 0$? The solution to this problem, as given by BS, is

$$\gamma > \frac{\pi}{2} - \arctan\frac{S-\tau+\tau\sqrt{1-S^2}}{\tau S+1-\sqrt{1-S^2}}. \tag{7}$$

where

$$\tau \equiv \frac{\sin 2\tilde{\beta}}{1-\sqrt{1-\sin^2 2\tilde{\beta}}}. \tag{8}$$

All the square roots in this paper are, by definition, positive.

In this paper I provide a simple derivation of the BS bound, which does not rely on any assumptions about the quark mixing matrix. I also consider the realistic situation where both S and C have been measured; this allows one to put a stronger bound on γ than when one knows only S, as was first pointed out by Botella and Silva [4]. Inspired by the result, quoted by BS, of a computation of z yielding

$$\arg z = 0.15 \pm 0.25\,, \tag{9}$$

I furthermore consider the situation where one assumes an upper bound on $|\arg z|$. Finally, I apply the BS bound, and its extensions, to the most recent measurements of S and C.

2 The Buchalla–Safir bound

I define

$$x \equiv \lambda \exp\left(2\mathrm{i}\tilde{\beta}\right) = \frac{\mathrm{e}^{-\mathrm{i}\gamma}+z}{\mathrm{e}^{\mathrm{i}\gamma}+z}. \tag{10}$$

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Then,

$$C = \frac{1 - |x|^2}{1 + |x|^2}, \tag{11}$$

and I furthermore define

$$I \equiv \frac{2\,\mathrm{Im}\,x}{1 + |x|^2}, \tag{12}$$

$$F \equiv \frac{|1 - x|^2}{1 + |x|^2} = 1 - \frac{2\,\mathrm{Re}\,x}{1 + |x|^2}. \tag{13}$$

Clearly,

$$0 \le F \le 2 \tag{14}$$

and

$$C^2 + I^2 + F^2 = 2\ F. \tag{15}$$

Solving (10) for z, one finds

$$z = -\cos\gamma + \frac{-I + \mathrm{i}C}{F}\ \sin\gamma. \tag{16}$$

This equation has an indeterminacy at the singular point $C = I = F = 0 \Leftrightarrow x = 1$, i.e. when $\sin\gamma = 0$, for arbitrary z.

From (16) it follows in particular that

$$F\,(\cos\gamma + \mathrm{Re}\,z) + I\sin\gamma = 0. \tag{17}$$

This equation has been first written down in [4]. It leads to the bound

$$|\mathrm{Re}\,z| \le \frac{\sqrt{F^2 + I^2}}{F}. \tag{18}$$

The solution to (17) may be written in the form

$$\gamma = \xi + \chi, \tag{19}$$

where (by definition)
(1) ξ is independent of $\mathrm{Re}\,z$, and
(2) $\chi = 0$ or $\chi = \pi$ when $\mathrm{Re}\,z = 0$.
One finds

$$\cos\xi = \frac{-I}{\sqrt{F^2 + I^2}}, \tag{20}$$

$$\sin\xi = \frac{F}{\sqrt{F^2 + I^2}}, \tag{21}$$

and

$$\sin\chi = \frac{F\,\mathrm{Re}\,z}{\sqrt{F^2 + I^2}}. \tag{22}$$

While ξ is perfectly defined by (20) and (21), χ as given by (22) suffers from the twofold ambiguity

$$\chi \to \pi - \chi. \tag{23}$$

Assuming, as Buchalla and Safir have done, that $\mathrm{Re}\,z > 0$, we see from (21) and (22) that both ξ and χ are angles either of the first or of the second quadrant. The Buchalla–Safir condition $\mathrm{Re}\,z > 0$ implies the lower bound on γ

$$\gamma > \xi = \arccos\frac{-I}{\sqrt{F^2 + I^2}}, \tag{24}$$

together with $\gamma < \xi + \pi$ too. Notice that

$$\mathrm{d}\xi = \frac{F\,\mathrm{d}I - I\,\mathrm{d}F}{F^2 + I^2}. \tag{25}$$

The inequality (24) provides a lower bound on γ but, unfortunately, one has to deal with discrete ambiguities. These occur because we are able to measure C but unable to measure I and F; rather, we only know $\sin 2\tilde\beta$ and S. Now,

$$I = \frac{2\,\mathrm{Re}\,\lambda}{1 + |\lambda|^2}\ \sin 2\tilde\beta + S\cos 2\tilde\beta, \tag{26}$$

$$F = 1 - \frac{2\,\mathrm{Re}\,\lambda}{1 + |\lambda|^2}\ \cos 2\tilde\beta + S\sin 2\tilde\beta. \tag{27}$$

Assuming that $\sin 2\tilde\beta$, S, and C are known, there is a fourfold ambiguity in I and F, since the signs of

$$\frac{2\,\mathrm{Re}\,\lambda}{1 + |\lambda|^2} = \pm\sqrt{1 - C^2 - S^2}, \tag{28}$$

$$\cos 2\tilde\beta = \pm\sqrt{1 - \sin^2 2\tilde\beta} \tag{29}$$

remain unknown. Using (25)–(29),

$$\frac{\mathrm{d}\xi}{\mathrm{d}C^2} = \frac{\left(-S - \sin 2\tilde\beta\right)\left(1 + |\lambda|^2\right)}{4\,(F^2 + I^2)\,\mathrm{Re}\,\lambda}. \tag{30}$$

Thus, given C, S, and $\sin 2\tilde\beta$, there are in reality four different angles ξ:
(1) ξ_1, in which both $\mathrm{Re}\,\lambda$ and $\cos 2\tilde\beta$ are positive,
(2) ξ_2, in which $\cos 2\tilde\beta$ is positive but $\mathrm{Re}\,\lambda$ is negative,
(3) ξ_3, in which both $\mathrm{Re}\,\lambda$ and $\cos 2\tilde\beta$ are negative, and
(4) ξ_4, in which $\mathrm{Re}\,\lambda$ is positive but $\cos 2\tilde\beta$ is negative.
Since F remains invariant, and I changes sign, when $\mathrm{Re}\,\lambda$ and $\cos 2\tilde\beta$ change sign simultaneously, we find that $\xi_3 = \pi - \xi_1$ and $\xi_4 = \pi - \xi_2$. From the assumption that $\mathrm{Re}\,z > 0$, and taking into account the indeterminacy in the signs of $\mathrm{Re}\,\lambda$ and $\cos 2\tilde\beta$, one can only deduce that γ must lie in between ξ_k and $\xi_k + \pi$ for all $k = 1, 2, 3$, and 4.

Let us now assume, with BS, the validity of the SM. Then $\cos 2\tilde\beta$ is positive and only the values ξ_1 and ξ_2 are allowed for ξ. This produces the lower bound

$$\gamma > \min\,(\xi_1, \xi_2)\,. \tag{31}$$

This lower bound is valid in the SM when C, S, and $\sin 2\tilde\beta$ are known. It still depends on C^2, since ξ_1 and ξ_2 contain $\sqrt{1 - C^2 - S^2}$. Consideration of (30), however, shows that, when $S > -\sin 2\tilde\beta$, ξ_1 decreases and ξ_2 increases with increasing C^2. Moreover, at the maximum allowed value of C^2, i.e. when $C^2 = 1 - S^2$, one has $\xi_1 = \xi_2$, since in general ξ_1 and ξ_2 only differ through the sign in front of $\sqrt{1 - C^2 - S^2}$, and that square root becomes zero when $C^2 = 1 - S^2$. This immediately leads to the BS bound: if $S > -\sin 2\tilde{\beta}$, then $\gamma > \xi_2\left(C^2 = 0\right)$. It can be shown [4] that, though different in appearance, this bound coincides with the one in (7).

One thus concludes that, if one assumes that $\cos 2\tilde{\beta} > 0$, then

$$\begin{cases} \gamma > \xi_2\left(C^2 = 0\right) \Leftarrow S > -\sin 2\tilde{\beta}, \\ \gamma > \xi_1\left(C^2 = 0\right) \Leftarrow S < -\sin 2\tilde{\beta}. \end{cases} \tag{32}$$

This may be put in a more transparent way if one defines

$$\varphi \equiv \frac{\arcsin S}{2}, \tag{33}$$

$$\alpha \equiv \pi - \tilde{\beta} - \gamma. \tag{34}$$

The lower bound on γ may then be rewritten as an upper bound on α:

$$\begin{cases} \alpha < \dfrac{\pi}{2} - \varphi \Leftarrow \varphi > -\tilde{\beta}, \\ \alpha < \pi + \varphi \ \Leftarrow \varphi < -\tilde{\beta}. \end{cases} \tag{35}$$

The discontinuity of the bound at $\varphi = -\tilde{\beta}$ should not come out as a surprise. The point $C = 0$, $S = -\sin 2\tilde{\beta}$ allows the singularity $C = I = F = 0$ referred to earlier. When $C = I = F = 0$, γ may be either 0 or π, independently of any assumption on z. Therefore no lower bound on γ may be derived if the experimentally allowed region for C and S includes that point.

It should be stressed that this derivation of the Buchalla–Safir bound on γ, or on α, contains basically no physical assumptions. Only (1)–(4) and (6), together with $\cos 2\tilde{\beta} > 0$ and $\mathrm{Re}\, z > 0$, are assumed. No assumptions are needed about the physics contained in the decay amplitudes, about the quark mixing matrix, or, indeed, about anything else; the sole crucial assumption is $\mathrm{Re}\, z > 0$. *The Buchalla–Safir bound is purely mathematical.*

I now return to the general case where one does not assume the SM. Then, γ may be either positive or negative and, from the assumption that $\mathrm{Re}\, z > 0$, it is only possible to produce a lower bound on $|\gamma|$, never on γ itself. Indeed, given the fourfold ambiguity in the determination of F and I, and the twofold ambiguity in the determination of χ – see (23) – there are eight solutions to (17) for γ. Since, when $\mathrm{Re}\, \lambda$ and $\cos 2\tilde{\beta}$ change sign simultaneously, I changes sign while F does not change, it is obvious from (17) that those eight solutions pair in four sets through the transformation $\gamma \to -\gamma$. Therefore, only a bound on $|\gamma|$ is possible. Now, computing

$$\tan^2 \xi_1\left(C^2 = 0\right) - \tan^2 \xi_2\left(C^2 = 0\right) = \frac{-4\sqrt{1 - S^2}\sqrt{1 - \sin^2 2\tilde{\beta}}}{\left(\sin 2\tilde{\beta} - S\right)^2}, \tag{36}$$

one finds that $\left|\tan \xi_1\left(C^2 = 0\right)\right|$ is always smaller than $\left|\tan \xi_2\left(C^2 = 0\right)\right|$. Hence,

$$|\gamma| > \arctan\left|\tan \xi_1\left(C^2 = 0\right)\right|. \tag{37}$$

Using again φ as defined in (33), one concludes that

$$|\gamma| > \left|\tilde{\beta} + \varphi\right|, \tag{38}$$

which is valid in any model provided $\mathrm{Re}\, z > 0$ – and provided the basic equations (1)–(4) and (6) hold, of course.

3 Assuming an upper limit on $|\arg z|$

In their work [3], Buchalla and Safir have quoted the result of a computation (in the context of the standard model) of z as yielding the result in (9). They have thereby justified their assumption $\mathrm{Re}\, z > 0$.[1] In this section I shall consider a different assumption,

$$|\cot \arg z| > L, \tag{39}$$

where L is some positive number. Clearly, this assumption is complementary to $\mathrm{Re}\, z > 0$; while $\mathrm{Re}\, z > 0$, by itself alone, leaves $\cot \arg z$ completely arbitrary, the condition (39), by itself alone, does not provide information on whether $\mathrm{Re}\, z$ is positive or negative. If L is, for instance, taken equal to 1, then (39) is well justified by (9).

In order to find the consequences of the assumption (39), I return to (16) and therefrom derive that

$$C \cot \arg z + F \cot \gamma + I = 0. \tag{40}$$

Hence,

$$|\cot \arg z| > L \Leftrightarrow \cot \gamma < \frac{-I - L\,|C|}{F} \quad \text{or} \quad \cot \gamma > \frac{-I + L\,|C|}{F}. \tag{41}$$

Clearly, this condition makes smaller the range for γ allowed by $\mathrm{Re}\, z > 0$ alone; that range, remember, is given by $\xi < \gamma < \xi + \pi$, where ξ belongs either to the first or to the second quadrant and $\cot \xi = -I/F$.

Let us now assume the validity of the SM. Then $\gamma \leq \pi - \beta$ and the relevant bound on γ following from (39) is the lower bound

$$\cot \gamma < \frac{-I - L\,|C|}{F} = \frac{\mp\sqrt{1 - C^2 - S^2}\sin 2\tilde{\beta} - S\cos 2\tilde{\beta} - L\,|C|}{1 \mp \sqrt{1 - C^2 - S^2}\cos 2\tilde{\beta} + S\sin 2\tilde{\beta}}. \tag{42}$$

This bound depends on the measured values of C, S, $\sin 2\tilde{\beta}$ and, besides, since $\cos 2\tilde{\beta}$ is positive in the SM, it depends on the sign multiplying $\sqrt{1 - C^2 - S^2}$.

[1] The assumption $\mathrm{Re}\, z > 0$ has also been recently used, and its validity scrutinized, in [5]. Of course, solid bounds on hadronic parameters are difficult or impossible to obtain from first principles, and the validity of calculations like the one yielding (9) is questionable.

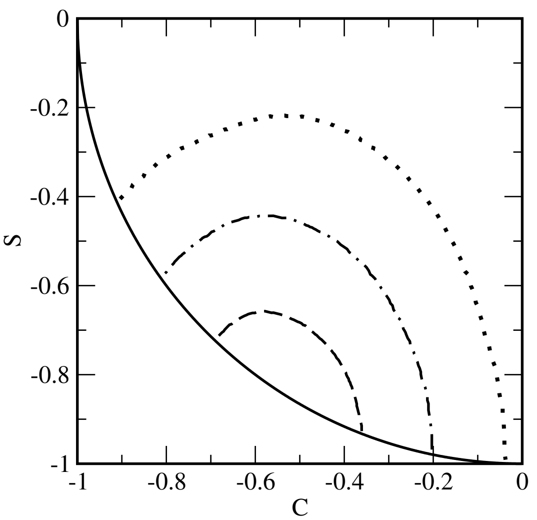

Fig. 1. The latest results of the Belle Collaboration for S and C. The full line bounds the circle defined by the condition $C^2 + S^2 \leq 1$. Within that circle, the dashed line bounds the region allowed by Belle at 68.3% C.L., the dot-dashed line bounds the region allowed at 95.45% C.L., and the dotted line bounds the region allowed at 99.73% C.L.

4 Application to the experimental results

The BS bound applies to the situation where S has been measured while C remains unknown but, in reality, both the Belle and BABAR Collaborations are able to measure S and C simultaneously and with comparable accuracy. Unfortunately, the latest results made public by the two groups do not quite coincide: while Belle [6] claims to have observed large CP violation in $B \to \pi^+\pi^-$, the BABAR measurements [7] are consistent with no CP violation at all. I shall apply the lower bound on γ given by the inequality (42) to the Belle results and, separately, to the "average" results of Belle and BABAR given by the Heavy Flavor Averaging Group [8]. I recall that, in inequality (42) one must use, for each pair of values for S and C, the sign in front of $\sqrt{1 - C^2 - S^2}$ yielding the less stringent bound. I shall assume fixed values for $\sin 2\tilde{\beta} = 0.736$ and $\cos 2\tilde{\beta} = \sqrt{1 - 0.736^2}$. For L I shall take the four values $L = 0$ – the case relevant for the BS bound, where $\mathrm{Re}\, z > 0$, but no lower bound on $|\cot \arg z|$, is assumed – and $L = \cot 0.9$, $\cot 0.65$, and $\cot 0.4$, corresponding to the 3σ, 2σ, and 1σ bounds, respectively, following from (9).

Belle [6] measures S and C to be both negative and not satisfying the constraint $S^2 + C^2 \leq 1$; enforcing the latest constraint, the Belle Collaboration has presented the allowed regions for C and S displayed in Fig. 1. The point $C = 0$, $S = -\sin 2\beta$ is disallowed at 99.9157% C.L., and therefore setting a BS lower bound on γ is possible. I performed scans of the allowed regions in the (C, S) plane advocated by the Belle Collaboration. For each value of the pair (C, S), and for each value of L, I computed the corresponding lower bound on γ. The results are the following. If one takes the 68.3% C.L. domain of Belle, then $\gamma > 21.8°$ if $L = 0$, $\gamma > 42.3°$ if $L = \cot 0.9$, $\gamma > 58.3°$ if $L = \cot 0.65$, and $\gamma > 93.6°$ if $L = \cot 0.4$. When one uses the region allowed by Belle at 95.45% C.L., one obtains $\gamma > 12.3°$ if $L = 0$, $\gamma > 24.1°$ if $L = \cot 0.9$, $\gamma > 33.9°$ if $L = \cot 0.65$, and $\gamma > 53.7°$ if $L = \cot 0.4$. Considering at last the 99.73% C.L. limits of Belle, one gets $\gamma > 3.6°$ if $L = 0$, $\gamma > 6.6°$ if $L = \cot 0.9$, $\gamma > 8.9°$ if $L = \cot 0.65$, and $\gamma > 12.5°$ if $L = \cot 0.4$; these very loose bounds reflect the proximity to this region of the point $C = 0$, $S = -\sin 2\beta$, for which no lower bound on γ is possible any more.

The Heavy Flavor Averaging Group has averaged the latest results made public by the Belle and BABAR Collabrations, and advocates [8]

$$\begin{aligned} S &= -0.61 \pm 0.14, \\ C &= -0.37 \pm 0.11. \end{aligned} \tag{43}$$

Accordingly, I shall use

$$\begin{aligned} S &\in [-0.75,\ -0.47]\,, \\ C &\in [-0.48,\ -0.26] \end{aligned} \tag{44}$$

at the 1σ level, and

$$\begin{aligned} S &\in [-0.89,\ -0.33]\,, \\ C &\in [-0.59,\ -0.15] \end{aligned} \tag{45}$$

at the 2σ level. The corresponding results are the following. If one uses the 1σ domains for S and C in (44), then $\gamma > 68.4°$ if $L = \cot 0.4$, $\gamma > 54.5°$ if $L = \cot 0.65$, $\gamma > 48.9°$ if $L = \cot 0.9$, and $\gamma > 31.1°$ if $L = 0$. If one uses the less stringent domains in (45), then $\gamma > 48.4°$ if $L = \cot 0.4$, $\gamma > 31.2°$ if $L = \cot 0.65$, $\gamma > 17.8°$ if $L = \cot 0.9$, and $\gamma > 8.6°$ if $L = 0$. The relevant bound on γ is in general obtained for the highest value of C and the lowest value of S in each domain, since that is the point closest to $C = 0$, $S = -\sin 2\beta$.

It is evident from the results above that assuming $|\cot \arg z| > L$, with a non-zero L, may greatly improve the lower bound on γ that one obtains from the BS condition $\mathrm{Re}\, z > 0$ alone.

5 Conclusions

I have shown that the Buchalla–Safir lower bound on γ is a purely mathematical consequence of the assumption $\mathrm{Re}\, z > 0$; the latter assumption follows from a computation of z within the standard model but, after that computation, the derivation of the BS bound itself requires no physics. I have improved the BS bound by assuming, above and beyond $\mathrm{Re}\, z > 0$, a lower bound on $|\cot \arg z|$. I have emphasized the fact that the presence, within the experimentally allowed region, of the point $(S, C) = (-\sin 2\beta, 0)$, prevents one from putting a lower bound on γ. I have applied the derived bounds to the (S, C) domains advocated by the most recent results made public by the Belle Collaboration and by the Heavy Flavor Averaging Group.

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